

AFOSR-TR- 78 - 0330

To appear in Ann. Statist., Sept. 1978  
issue.

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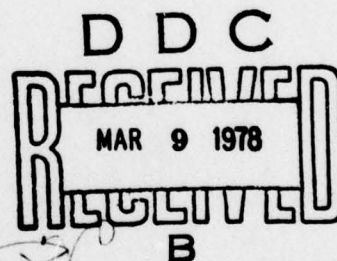
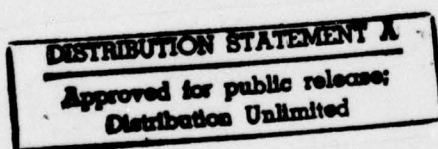
Positive Dependence of the Bivariate and  
Trivariate Absolute Normal,  $t$ ,  $\chi^2$ , and  $F$   
Distributions

by

M. Abdel-Hameed  
University of North Carolina at Charlotte

and

Allan R. Sampson  
Abbott Laboratories  
and  
Florida State University



November, 1977  
Department of Statistics  
Florida State University  
Tallahassee, Florida 32306

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M. Abdel-Hameed<sup>1</sup>  
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ABSTRACT

It is shown that the bivariate density of the absolute normal distribution is totally positive of order 2. Necessary and sufficient conditions are given for the trivariate density of the absolute normal distribution to be totally positive of order 2 in pairs of arguments. These results are then used to show that certain generalized bivariate and trivariate  $t$ ,  $\chi^2$  and F random variables are associated.

*NC* <sup>1</sup>The research of this author is supported in part by the Air Force Office of Scientific Research, AFSC, USAF, under grant AFOSR 76-2999 and the Office of Naval Research Contract N00014-76-C-0840.

*FZ* <sup>2</sup>The research of this author is supported in part by the Office of Naval Research under grant N00014-67-A-0235-0006.

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AMS 1970 Subject Classifications: 62H05.

Key Words: Total Positivity, positive quadrant dependence, conditionally increasing in sequence, association, multivariate  $t$  distribution, multivariate F distribution, multivariate normal distribution.

1. Introduction. Motivated by needs in simultaneous inference, numerous authors have established inequalities for joint probabilities in terms of marginal probabilities. Typically in these inequalities the underlying random variables are jointly normal and most of the proofs are of an analytic nature. In this paper we obtain stronger dependence results in the bivariate and trivariate cases by using certain notions of multivariate dependence.

Suppose  $(X_1, \dots, X_p)' \sim N_p(\underline{0}, \underline{\Sigma})$ , where  $N_p(\underline{0}, \underline{\Sigma})$  denotes the law of a  $p$ -variate normal random vector with mean  $\underline{0}$  and nonsingular covariance matrix  $\underline{\Sigma} = \{\rho_{ij}\sigma_i\sigma_j\}$ . For  $i = 1, \dots, n$ , let  $\underline{z}_i \equiv (z_{1i}, \dots, z_{pi})' \sim N_p(\underline{0}, \underline{\psi}_i)$ , where  $z_1, \dots, z_n$  are independent random variables. Further, for  $i = 1, \dots, p$ , let  $T_1^i, \dots, T_{q_i}^i$  be independently and identically distributed according to  $N(0, \sigma_{T_i}^2)$ . Now assume  $(X_1, \dots, X_p)'$ ,  $\{\underline{z}_i\}$ ,  $\{T_k^1\}_{k=1}^{q_1}, \dots, \{T_k^p\}_{k=1}^{q_p}$  are mutually independent sets of random variables.

Define

$$(1.1) \quad S_k^2 = \sum_{\ell=1}^n z_{k\ell}^2 / (\psi_\ell)_{kk}, k = 1, \dots, p,$$

and

$$(1.2) \quad S_k^{*2} = \sum_{\ell=1}^{q_k} (T_\ell^k)^2 / \sigma_{T_k}^2, k = 1, \dots, p.$$

DEFINITION. (Lehmann [1966]). The random variables  $U_1, \dots, U_\alpha$  are positively quadrant dependent (PQD) if  $P[n(U_i \leq u_i)] \geq \prod P[U_i \leq u_i]$ , for all real numbers  $u_1, \dots, u_\alpha$ .

In the case  $p = 2$ , i.e., the bivariate case, Khatri [1967] showed that  $|X_1|, |X_2|$  are PQD and that  $S_1^2, S_2^2$  are PQD. Šidák [1967, 1971] proved that  $|X_1|/S_1, |X_2|/S_2$  are PQD. Halperin [1967] obtained the slightly stronger result that  $|X_1|/(S_1^2 + S_1^{*2})^{\frac{1}{2}}, |X_2|/(S_2^2 + S_2^{*2})^{\frac{1}{2}}$  are PQD. Dunn 1958 had previously obtained similar results.

For the  $p = 3$  case, similar results hold. Khatri showed that  $|X_1|, |X_2|, |X_3|$  are PQD if  $\Sigma$  is of the form  $\{\beta_i \beta_j\}$ ,  $i \neq j$ . Khatri also showed under this condition that  $S_1^2, S_2^2, S_3^2$  are PQD. Sidák [1971] proved that  $|X_1|/S_1, \dots, |X_3|/S_3$  are PQD if the correlation between  $X_i$  and  $X_j$  is of the form  $\lambda_i \lambda_j \rho_{ij}$  ( $i, j = 1, \dots, 3; i \neq j$ ),  $|\lambda_i| \leq 1$  ( $i = 1, 2, 3$ ),  $\{\rho_{ij}\}$  is any fixed correlation matrix; and if the correlation between  $z_{\ell i}$  and  $z_{ki}$  is of the form  $\tau_{\ell i} \tau_{ki}$  ( $\ell, k = 1, 2, 3; \ell \neq k$ ,  $i = 1, \dots, n$ ) where  $|\tau_{\ell i}| < 1$  ( $\ell = 1, 2, 3; i = 1, \dots, n$ ).

Some results have been obtained for higher dimensions by the above authors.

Note that up to constants  $(X_1/S_1, \dots, X_p/S_p)'$  is a multivariate Student's t-random vector (considered in the bivariate case by Siddiqui [1967]);  $(X_1/S_1, \dots, X_p/S_p)'$  is a generalized multivariate Student's t-random vector (Sidák [1971]); and  $(S_1^2, \dots, S_p^2)'$  is a multivariate  $\chi^2$  random vector (Krishnamoorthy and Parthasarathy [1951], Jensen [1970]). Jogdeo [1977] defined a class of multivariate random variables called "contaminated random variables" and showed that their absolute values are associated. It is worth noting that the multivariate normal distributions discussed in Khatri and Sidák papers mentioned above can be viewed as "contaminated random variables" as shown on p. 498 of Jogdeo's paper. Pitt [1977] strengthened the results of Khatri [1967] and Sidák [1967, 1971] in the bivariate case, and proved that if  $n(x_1, x_2)$  is the standard normal density on  $R^2$  and if  $A = -A$  and  $B = -B$  are convex subsets of  $R^2$ , then  $P[(x_1, x_2) \in A \cap B] \geq P[(x_1, x_2) \in A] P[(x_1, x_2) \in B]$ . Dykstra and Hewett [1977] established positive dependence of the roots of a Wishart matrix.

The preceding results were derived basically independently of each other and each proof involved analytic techniques specific to that result. In this paper we obtain the following basic results: (a) the density of  $|X_1|, |X_2|$  is totally positive of order 2 (b) a necessary and sufficient condition that  $|X_1|, |X_2|, |X_3|$  be totally positive of order 2 in pairs of arguments is that  $\prod_{i < j} \text{sgn}(\lambda_{ij}) \leq 0$ , where  $\underline{\lambda} \equiv \{\lambda_{ij}\} = \underline{\Sigma}^{-1}$ ; (c)  $S_1^2 + S_1^{*2}, S_2^2 + S_2^{*2}, S_3^2 + S_3^{*2}$ , are associated random variables and that  $|X_1|/(S_1^2 + S_1^{*2})^{\frac{1}{2}}, |X_2|/(S_2^2 + S_2^{*2})^{\frac{1}{2}}, |X_3|/(S_3^2 + S_3^{*2})^{\frac{1}{2}}$  are associated random variables. (The same results hold for  $p = 2$ .)

2. Total Positivity of the Bivariate Absolute Normal. We employ the following definitions and implications.

DEFINITION 2.1. (Karlin [1968]). A function  $f: \mathbb{R}^2 \rightarrow [0, \infty)$  is totally positive of order 2 ( $TP_2$ ) if the second order determinant  $\det\{f(u_i, v_j)\}$  is nonnegative for each choice  $u_1 < u_2, v_1 < v_2$ .

DEFINITION 2.2. (Esary, Proschan and Walkup [1967]). The random variables  $U_1, \dots, U_\alpha$  are associated if  $\text{Cov}[f(U_1, \dots, U_\alpha), g(U_1, \dots, U_\alpha)] \geq 0$  for all nondecreasing functions  $f, g$ .

DEFINITION 2.3. (Barlow and Proschan [1975]). Let  $\alpha$  be an integer exceeding 2. A function  $f: \mathbb{R}^\alpha \rightarrow [0, \infty)$  is said to be totally positive of order 2 in pairs ( $TP_2$  in pairs) if for any pair of arguments  $u_a, u_b$ ,  $f(u_1, \dots, u_a, \dots, u_b, \dots, u_\alpha)$ , viewed as a function of  $u_a, u_b$  with remaining arguments fixed, is  $TP_2$ .

DEFINITION 2.4. (Barlow and Proschan [1975]). The random variables  $U_1, \dots, U_\alpha$  are conditionally increasing in sequence if for  $i = 1, \dots, \alpha$   $P(U_i > u_i / U_{i-1} = u_{i-1}, \dots, U_1 = u_1)$  is increasing in  $u_1, \dots, u_{i-1}$ .

For  $s > 0$ , let

$$\begin{aligned} \gamma^{(s)}(t) &= (t)^{s-1}/\Gamma(s), \quad t \geq 0 \\ &= 0, \quad t < 0. \end{aligned}$$

For  $m > 0, n > 0$ , define

$$\psi_{m,n}(u_1, u_2) = E[\gamma^{(m)}(u_1 - u_1) \gamma^{(n)}(u_2 - u_2)],$$

where the expectation in the right hand side is taken with respect to the joint distribution of  $U_1$  and  $U_2$ .

DEFINITION 2.5. (Shaked [1977]). Two random variables  $U_1$  and  $U_2$  are said to be dependent by total positivity of order two with degree  $(m,n)$  (denoted by  $DTP(m,n)$ ) if  $\psi_{m,n}(u_1, u_2)$  is  $TP_2$  in  $u_1$  and  $u_2$ .

The following is an appropriate extension of  $DTP(m,n)$  for more than two random variables.

DEFINITION 2.6. The random variables  $U_1, \dots, U_\alpha$  are said to be dependent by total positivity of order two with degree  $(m,n)$  in pairs (denoted by  $DTP(m,n)$  in pairs) if for every pair of arguments  $u_a, u_b$ ,

$$\psi_{m,n}(u_a, u_b) \stackrel{\text{def}}{=} E[\{\gamma^{(m)}(U_a - u_a) \gamma^{(n)}(U_b - u_b)\} / U_i, i = 1, \dots, \alpha, i \neq a, b]$$

is  $TP_2$  in  $u_a, u_b$ .

The following lemma is closely related to Theorem 4.2, p. 143, of Barlow and Proschan [1975] and Proposition 3.4 of Shaked [1977].

LEMMA 2.1. Let the random variables  $U_1, \dots, U_\alpha$  have joint density

$f_{U_1, \dots, U_\alpha}(u_1, \dots, u_\alpha)$ . Then the following implications hold:

$f_{U_1, \dots, U_\alpha}(u_1, \dots, u_\alpha)$  is  $TP_2$  in pairs  $\iff U_1, \dots, U_\alpha$  are  $DTP(0,0)$  in

pairs  $\implies U_1, \dots, U_\alpha$  are conditionally increasing in sequence  $\implies U_1, \dots, U_\alpha$

are associated  $\implies U_1, \dots, U_\alpha$  are PQD.

A more detailed examination of  $DTP(m,n)$  in pairs and its relationship to the dependence concepts given by Alam and Wallenius [1976], Esary and Proschan [1972] and Shaked [1977] are being currently examined by the authors and will appear in the future. The implications given in lemma 2.1 are, however, sufficient for our purpose.

In order to obtain our main bivariate result, we require the following lemma.

LEMMA 2.2. Let  $f(u,v) = k_1(u) k_2(v) g(uv)$  for  $u \geq 0, v \geq 0$  and  $f(u,v) = 0$ , otherwise. Assume  $k_1 \geq 0, k_2 \geq 0$ , and  $g \geq 0$ . If  $g$  is nondecreasing and  $\ln g$  is convex, then  $f$  is  $TP_2$ .

PROOF. Since  $f = 0$  for  $u < 0$  or  $v < 0$ , it suffices to consider  $0 \leq u_1 < u_2, 0 \leq v_1 < v_2$ , in showing  $\det\{f(u_i, v_j)\} \geq 0$ . Note that  $\det\{f(u_i, v_j)\} = \prod_{i=1}^2 (k_1(u_i) k_2(v_j)) \det\{g(u_i v_j)\}$ , and thus we need only to show that  $\det\{g(u_i v_j)\} \geq 0$ . Define  $t_1 = u_1 v_1, t_1 + \Delta_1 = u_2 v_1, t_2 = u_1 v_2, t_2 + \Delta_2 = u_2 v_2$ , so that  $0 \leq \Delta_1 < \Delta_2$ . Observe that

$$\begin{aligned} \det\{g(u_i v_j)\} &= g(t_1)g(t_2 + \Delta_2) - g(t_1 + \Delta_1)g(t_2) \\ &\geq g(t_1)g(t_2 + \Delta_1) - g(t_1 + \Delta_1)g(t_2) \\ &\geq 0, \end{aligned}$$

where the first inequality follows because  $g \geq 0$  and nondecreasing and the second inequality because  $g$  is logarithmically convex.

THEOREM 2.1. Let  $(X_1, X_2)' \sim N_2(0, \Sigma)$ . Then the joint density function  $f_{|X_1|, |X_2|}(x_1, x_2)$ , of  $|X_1|, |X_2|$  is  $TP_2$ .

PROOF. For  $x_1 < 0$  or  $x_2 < 0$ ,  $f_{|x_1|, |x_2|}(x_1, x_2) = 0$ , and for  $x_1 \geq 0$ ,  $x_2 \geq 0$  it is readily shown that

$$f_{|x_1|, |x_2|}(x_1, x_2) = k_1(x_1)k_2(x_2)g(x_1x_2),$$

where

$$k_i(s) = \exp[-s^2/\theta_i^2], \quad i = 1, 2,$$

$$g(s) = 4c \cosh(\rho s/(\theta_1\theta_2))$$

and

$$\theta_i = (2 - 2\rho^2)^{\frac{1}{2}}\sigma_i, \quad i = 1, 2; \quad c^{-1} = 2\pi\sigma_1\sigma_2(1 - \rho^2)^{\frac{1}{2}}$$

Straightforward calculations yield that  $g$  is nondecreasing and logarithmically convex, so that Lemma 2 immediately yields that  $f_{|x_1|, |x_2|}(x_1, x_2)$  is  $TP_2$ .  $\square$

REMARK 2.1. From Lemma 2.1, it follows that the random variables  $|X_1|, |X_2|$  are conditionally increasing in sequence, associated, and PQD.

3. Total Positivity of the Trivariate Absolute Normal. In this section we give a necessary and sufficient condition for the density function of the trivariate absolute normal variable to be  $TP_2$  in pairs. In section 4 we use this result to show that a trivariate  $\chi^2$  and a trivariate  $t$ -distribution are associated and, hence, are PQD.

Let  $(X_1, X_2, X_3)' \sim N_3(0, \Sigma)$  have a trivariate normal distribution with mean  $0$  and covariate matrix  $\Sigma$ . Let  $\Lambda \equiv \{\lambda_{ij}\} = \Sigma^{-1}$ . Then the joint p.d.f.,  $f_{|x_1|, |x_2|, |x_3|}(x_1, x_2, x_3)$ , of  $|X_1|, |X_2|, |X_3|$ , for  $(x_1, x_2, x_3)$  in the positive octant is given by

$$(3.1) \quad f_{|x_1|, |x_2|, |x_3|}(x_1, x_2, x_3) = K_{\Lambda} \exp[-\frac{1}{2}(\lambda_{11}x_1^2 + \lambda_{22}x_2^2 + \lambda_{33}x_3^2)]g(x_1, x_2, x_3),$$

where

$$K_{\Lambda} = 2(\sqrt{2\pi})^{-3} |\Lambda|^{\frac{1}{2}},$$

and

$$g(x_1, x_2, x_3) = \sum_{i=0}^1 \sum_{j=0}^1 \exp[(-1)^i \lambda_{12} x_1 x_2 + (-1)^j \lambda_{13} x_1 x_3 + (-1)^{i+j+1} \lambda_{23} x_2 x_3].$$

The density is 0, otherwise.

Hence, to show that  $f_{|X_1|, |X_2|, |X_3|}(x_1, x_2, x_3)$  is  $TP_2$  in pairs it suffices to show that  $g(x_1, x_2, x_3)$  is  $TP_2$  in pairs. To do so we require the following two lemmas whose proofs are straightforward.

**LEMMA 3.1.** Let  $\Lambda_0$  be a fixed  $3 \times 3$  positive definite matrix and define  $D_e$  as a diagonal matrix with elements  $\pm 1$ . Then the p.d.f.  $f_{|X_1|, |X_2|, |X_3|}(x_1, x_2, x_3)$  given by (3.1), viewed as a function of  $\Lambda$ , is invariant on the set  $\{\Lambda: \Lambda = D_e \Lambda_0 D_e\}$ .

Define  $\text{sgn}(x) = 1$  if  $x > 0$ ;  $= 0$  if  $x = 0$ ;  $= -1$  if  $x < 0$ .

**LEMMA 3.2.** A necessary and sufficient condition that there exists  $D_e$ , a diagonal matrix with elements  $\pm 1$ , so that the off-diagonal elements of  $D_e \Lambda_0 D_e$  are all negative (positive) is that  $\prod_{i < j} \text{sgn}(\lambda_{ij}^0) = -1$  ( $= 1$ ), where  $\lambda_{ij}^0$  is the  $i, j^{\text{th}}$  element of  $\Lambda_0$ .

**THEOREM 3.1.** Let  $(X_1, X_2, X_3)' \sim N_3(0, \Sigma)$ . Then a necessary and sufficient condition that the joint density function  $f_{|X_1|, |X_2|, |X_3|}(x_1, x_2, x_3)$  of  $|X_1|, |X_2|, |X_3|$  be  $TP_2$  in pairs is that  $\prod_{i < j} \text{sgn}(\lambda_{ij}) \leq 0$ , where  $\Lambda \equiv \{\lambda_{ij}\} = \Sigma^{-1}$ .

**PROOF.**

**SUFFICIENCY.** If  $\prod_{i < j} \text{sgn}(\lambda_{ij}) = -1$ , then by Lemmas 3.1 and 3.2, we may suppose that  $-\lambda_{12} > 0, -\lambda_{13} > 0, -\lambda_{23} > 0$ . Let

$$(3.2) \quad u = -\lambda_{13} x_1 x_3,$$

$$v = -\lambda_{23} x_2 x_3,$$

and

$$\alpha = -\lambda_{12}/(\lambda_{13}\lambda_{23}x_3^2),$$

so that  $-\lambda_{12}x_1x_2 = \alpha uv$ . Without loss of generality, we only show

$f_{|x_1|, |x_2|, |x_3|}(x_1, x_2, x_3)$  is  $TP_2$  in  $x_1 > 0, x_2 > 0$  for  $x_3 > 0$  fixed.

This is equivalent to showing that for  $\alpha > 0$   $h_\alpha(u, v)$  is  $TP_2$  for  $u > 0, v > 0$ , where

$$\begin{aligned} (3.3) \quad h_\alpha(u, v) &= P_\alpha(u, v) + P_\alpha(u, -v) + P_\alpha(-u, v) + P_\alpha(-u, -v), \\ &= 2e^{\alpha uv} \cosh(u + v) + 2e^{-\alpha uv} \cosh(u - v) \end{aligned}$$

and

$$P_\alpha(u, v) = \exp[u + v + \alpha uv].$$

Let

$$(3.4) \quad \Delta_\alpha(u, v) \equiv h_\alpha(u, v) \frac{\partial^2 h_\alpha(u, v)}{\partial u \partial v} - \frac{\partial h_\alpha(u, v)}{\partial u} \frac{\partial h_\alpha(u, v)}{\partial v}.$$

To verify that  $h_\alpha(u, v)$  is  $TP_2$ , we verify for  $u > 0, v > 0, \alpha > 0$  that

$$(3.5) \quad \Delta_\alpha(u, v) \geq 0.$$

(see Karlin [1968], p. 49). Direct calculation yields that

$$\begin{aligned} \frac{\partial h_\alpha(u, v)}{\partial u} &= (1 + \alpha v)[P_\alpha(u, v) - P_\alpha(-u, v)] + \\ &\quad (1 - \alpha v)[P_\alpha(u, -v) - P_\alpha(-u, -v)], \end{aligned}$$

$$\begin{aligned} \frac{\partial h_\alpha(u, v)}{\partial v} &= (1 + \alpha u)[P_\alpha(u, v) - P_\alpha(u, -v)] + \\ &\quad (1 - \alpha u)[P_\alpha(-u, v) - P_\alpha(-u, -v)], \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 h_\alpha(u, v)}{\partial u \partial v} &= \alpha[P_\alpha(u, v) + P_\alpha(-u, -v) - P_\alpha(-u, v) - P_\alpha(u, -v)] \\ &\quad + (1 + \alpha v)(1 + \alpha u)P_\alpha(u, v) \\ &\quad - (1 - \alpha u)(1 + \alpha v)P_\alpha(-u, v) - (1 + \alpha u)(1 - \alpha v)P_\alpha(u, -v) \\ &\quad + (1 - \alpha u)(1 - \alpha v)P_\alpha(-u, -v), \end{aligned}$$

so that after simplification, we have

$$(3.6) \quad \Delta_{\alpha}(u,v) = 2\alpha[e^{2\alpha uv} \cosh(2(u+v)) - e^{-2\alpha uv} \cosh(2(u-v))] \\ + 4(2 + \alpha) \sinh(2\alpha uv) + 8\alpha v [\sinh(2u) + \alpha u \cosh(2u)] \\ + 8\alpha u [\sinh(2v) + \alpha v \cosh(2v)].$$

The first term of (3.6) is nonnegative by the monotonicity of  $e^t$  and the monotonicity of  $\cosh(|t|)$ . The remaining 3 terms of (3.4) are nonnegative because  $\cosh t \geq 0$ , and  $\sinh t \geq 0$  for  $t \geq 0$ . Thus (3.5) holds.

If  $\prod_{i < j} \text{sgn}(\lambda_{ij}) = 0$ , then either two or more of the  $\lambda_{ij}$ 's equal to zero, or exactly one of the  $\lambda_{ij}$ 's equal to zero. The case where two or more of the  $\lambda_{ij}$ 's equal to zero follows from the bivariate case discussed in Section 2. If exactly one of the  $\lambda_{ij}$ 's equals to zero, say  $\lambda_{12}$ , then in Equation (3.2) divide by the other two  $\lambda_{ij}$ 's, so that  $\alpha = 0$  and then apply a technique similar to the one used when  $\alpha > 0$  to show that the density is  $TP_2$  in pairs for fixed  $x_3$ . In this case, to show  $TP_2$  in pairs for fixed  $x_1$  or  $x_2$  the argument would reduce to the bivariate case argument.

NECESSITY. Suppose  $\prod_{i < j} \text{sgn}(\lambda_{ij}) = 1$ , so that by Lemmas 3.1 and 3.2 we can assume  $\lambda_{12} > 0$ ,  $\lambda_{13} > 0$ ,  $\lambda_{23} > 0$ . Define  $u, v, \alpha$  as in the proof of the sufficiency, but note  $u, v$ , and  $\alpha < 0$ . We proceed to show that there exists  $x_3 > 0$  so that  $f_{|x_1|, |x_2|, |x_3|}(x_1, x_2, x_3)$  has negative second order determinant for certain  $x_1 > 0, x_2 > 0$ . To do this, we let  $x_3 = [\lambda_{12}/(\lambda_{13}\lambda_{23})]^{\frac{1}{2}}$ , so that  $\alpha = -1$ , and then show that there exists an open set so that  $\Delta_{-1}(u, v)$  defined in (3.4) is negative. To find such an open set, we show that there exists  $t < 0$  so that  $\Delta_{-1}(t, t)$  is negative and then appeal to the continuity of  $\Delta_{-1}(u, v)$ . Note that

$$\Delta_{-1}(t, t) = 2e^{-2t^2} [1 - \cosh(4t)] - 16t [\sinh(2t) - t \cosh(2t)].$$

Observe that  $\cosh(2t) \geq 1$  and that for suitably small negative  $t$ ,  $\sinh(2t) - t \cosh(2t) < 0$ , so that for suitably small negative  $t$ ,  $\Delta_1(t, t) < 0$ . Hence, we can conclude that

$f_{|x_1|, |x_2|, |x_3|}(x_1, x_2, x_3)$  is not  $TP_2$  in pairs if  $\prod_{i < j} \text{sgn}(\lambda_{ij}) = 1$ .  $\square$

REMARK 3.1. If  $\prod_{i < j} \text{sgn}(\lambda_{ij}) \leq 0$ , then, using Lemma 2.1 and Theorem 3.1, we have that the random variables  $|x_1|, |x_2|, |x_3|$  are conditionally increasing in sequence, and associated.

For the general multivariate normal case without absolute values, we note that Barlow and Proschan [1975, Chapter 4] proved that the multivariate normal density function is  $TP_2$  in pairs if and only if  $\lambda_{ij} \leq 0$  for  $i \neq j$  where  $\Lambda = \Sigma^{-1}$ .

#### 4. The Association of Bivariate and Trivariate $\chi^2$ and $t$ Distributions.

In this section we use the results of the previous sections to obtain the association of certain bivariate, and trivariate,  $\chi^2$ ,  $t$ , and  $F$  distributions.

To prove the results of this section, we make use of the following two lemmas which by themselves are quite interesting and useful. Lemma 4.1 is a special case of Theorem 4.1 of Jogdeo [1977].

LEMMA 4.1. Let  $U_1, \dots, U_\alpha$  be positive random variables. If  $U_1, \dots, U_\alpha$  are associated, then  $U_1^{-1}, \dots, U_\alpha^{-1}$  are associated.

LEMMA 4.2. Suppose that the nonnegative random variables  $U_1, \dots, U_\alpha$  are independent of the nonnegative random variables  $V_1, \dots, V_\alpha$ . If  $U_1, \dots, U_\alpha$  are associated and  $V_1, \dots, V_\alpha$  are associated, then  $U_1 V_1, \dots, U_\alpha V_\alpha$  are associated.

THEOREM 4.1. (a) For  $p = 2$ ,  $S_1^2 + S_1^{*2}$ ,  $S_2^2 + S_2^{*2}$  are associated random variables.

(b) For  $p = 3$ , if  $\prod_{k < j} \text{sgn}(\psi_i^{-1})_{kj} \leq 0$ ,  $i = 1, \dots, n$ , then  $S_1^2 + S_1^{*2}$ ,  $S_2^2 + S_2^{*2}$  and  $S_3^2 + S_3^{*2}$  are associated random variables, where  $(\psi_i^{-1})_{kj}$  denotes  $k, j^{\text{th}}$  element of  $\psi_i^{-1}$ .

PROOF of (a): By Lemma 2.1, Theorem 2.1 and the invariance of association under nondecreasing transformations ( $P_4$  of [4]) we have, for  $i = 1, \dots, n$ , that  $Z_{1j}^2, Z_{2i}^2$  are associated. Because  $\{Z_i\}$ ,  $S_1^{*2}, S_2^{*2}$  are independent, we have that  $Z_{11}^2, Z_{21}^2, \dots, Z_{1n}^2, Z_{2n}^2, S_1^{*2}, S_2^{*2}$  are associated ( $P_2$  of [4]). Since  $S_1^2 + S_1^{*2}, S_2^2 + S_2^{*2}$  are nondecreasing functions of the  $Z_{ij}^2, S_1^{*2}, S_2^{*2}$ , we obtain  $S_1^2 + S_1^{*2}, S_2^2 + S_2^{*2}$  are associated.

PROOF of (b): Using Theorem 3.1 and Lemma 2.1, we can prove (b) in a similar fashion to (a) with the obvious modifications.

REMARK 4.1. Note that Khatri's condition that  $\psi_i$  is of the form  $(\beta_i \beta_j)$   $|\beta_i| < 1$ ,  $i = 1, 2, 3$  implies that  $\prod_{k < j} \text{sgn}(\psi_i^{-1})_{kj} \leq 0$ .

COROLLARY 4.1. (a) For  $p = 2$ ,  $(S_1^2 + S_1^{*2})^{-\frac{1}{2}}, (S_2^2 + S_2^{*2})^{-\frac{1}{2}}$  are associated random variables.

(b) For  $p = 3$ , if  $\prod_{k < j} \text{sgn}(\psi_i^{-1})_{kj} \leq 0$ ,  $i = 1, \dots, n$ , then  $(S_1^2 + S_1^{*2})^{-\frac{1}{2}}, (S_2^2 + S_2^{*2})^{-\frac{1}{2}}$ , and  $(S_3^2 + S_3^{*2})^{-\frac{1}{2}}$  are associated random variables.

PROOF: The proof follows from Theorem 4.1 and the square root analogue of Lemma 4.1.  $\square$

THEOREM 4.2. (a) For  $p = 2$ , the random variables  $|X_1|/(S_1^2 + S_1^{*2})^{\frac{1}{2}}$  and  $|X_2|/(S_2^2 + S_2^{*2})^{\frac{1}{2}}$  are associated.

(b) For  $p = 3$ , if  $\prod_{i < j} \text{sgn}(\lambda_{ij}) \leq 0$ ,  
 $\prod_{k < j} \text{sgn}(\psi_i^{-1})_{kj} \leq 0$ ,  $i = 1, 2, \dots, n$ , then  $|x_1|/(S_1^2 + S_1^{*2})^{\frac{1}{2}}$ ,  $|x_2|/(S_2^2 + S_2^{*2})^{\frac{1}{2}}$ ,  
 $|x_3|/(S_3^2 + S_3^{*2})^{\frac{1}{2}}$  are associated random variables.

PROOF: The proof of the theorem follows from Theorem 2.1, Theorem 3.1, Lemma 4.2, and Corollary 4.1.  $\square$

Up to constants, a bivariate and a trivariate F random vector can be defined by:

$$\tilde{F}_{(2)} = (S_1^2/S_1^{*2}, S_2^2/S_2^{*2})',$$

and

$$\tilde{F}_{(3)} = (S_1^2/S_1^{*2}, S_2^2/S_2^{*2}, S_3^2/S_3^{*2})'.$$

THEOREM 4.3. (a)  $S_1^2/S_1^{*2}, S_2^2/S_2^{*2}$  are associated random variables

(b) If  $\prod_{k < j} \text{sgn}(\psi_i^{-1})_{kj} \leq 0$ ,  $i = 1, \dots, n$ , then the random variables  $S_1^2/S_1^{*2}, S_2^2/S_2^{*2}, S_3^2/S_3^{*2}$  are associated.

PROOF: The proof of the theorem follows immediately from the proof of Theorem 4.1, Lemma 4.1 and Lemma 4.2.

REMARK 4.2. Theorems 4.1, 4.2, 4.3 and Corollary 4.1 remain true as long as  $S_1^{*2}$  and  $S_2^{*2}$  are any pair of positive independent random variables such that  $(x_1, \dots, x_p)', \{Z_i\}, S_1^{*2}, S_2^{*2}$  are all mutually independent sets of random variables.

We conclude this paper with the following conjecture for the  $TP_2$  in pairs of the multivariate absolute normal, of dimension larger than 3.

CONJECTURE. Let  $f_{|x_1|, \dots, |x_p|}(x_1, \dots, x_p)$  be the p.d.f. of the multivariate absolute normal,  $p > 3$ . A necessary and sufficient condition for it to be  $TP_2$  in pairs is that there exists  $\underline{D}_e$ , a diagonal matrix with elements  $\pm 1$ , such that the off-diagonal elements of  $\underline{D}_e \underline{\Sigma}^{-1} \underline{D}_e$  are all negative.

Note that if this conjecture were true, then the corresponding result concerning the multivariate t-distribution could be proved directly in the same fashion as Theorem 4.2

Acknowledgement. After completion of this research, it was brought to our attention that Professor F. Proschan in a private communication independently obtained Theorem 2.1. His method of proof is different from ours. Also the authors wish to thank Professor P. E. Lin for helpful conversations.

We are grateful to the referee for careful reading of the paper and his valuable suggestions. He brought to our attention that Lemma 4.1 is a special case of Theorem 4.1 of Jogdeo [1977].

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4. TITLE (and Subtitle) 6 POSITIVE DEPENDENCE OF THE BIVARIATE AND TRIVARIATE ABSOLUTE NORMAL, t, X SQUARED AND F DISTRIBUTIONS.		5. TYPE OF REPORT & PERIOD COVERED 9 Interim rept.	
7. AUTHOR(s) 10 M./Abdel-Hameed Allan R. Sampson		8. CONTRACT OR GRANT NUMBER(s) 15 AFOSR-76-29994	
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of North Carolina-Charlotte Department of Mathematics Charlotte, NC 28223		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 16 61102F 17 2304 A5	
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM Bolling AFB, DC 20332		12. REPORT DATE 11 Nov 77 November 1977	
		13. NUMBER OF PAGES 15 12 19p.	
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED	
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE	
16. DISTRIBUTION STATEMENT (of this Report)  Approved for public release; distribution unlimited			
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)			
18. SUPPLEMENTARY NOTES			
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)			
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)  It is shown that the bivariate density of the absolute normal distribution is totally positive of order 2. Necessary and sufficient conditions are given for the trivariate density of the absolute normal distribution to be totally positive of order 2 in pairs of arguments. These results are then used to show that certain generalized bivariate and trivariate t, $X^2$ and F random variables are associated. X-Squared			

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